Finite Math - Spring 2017

Lecture Notes - 2/24/2017

Homework

- Section 3.4 - 7, 11, 12, 15, 18, 27, 30, 33, 47

Section 3.4 - Present Value of an Annuity; Amortization

Present Value of an Annuity. In the next concept, we will look at making a large deposit in order to have a fund which we can make constant withdraws from. We make an initial deposit, then make withdraws at the end of each interest period. We should have a balance of \$0 at the end of the predetermined amount of time the fund should last.

Example 1. *How much should you deposit into an account paying 6% compounded semiannually in order to be able to withdraw \$2000 every 6 months for 2 years? (At the end of the 2 years, there should be a balance of \$0 in the account.)*

Solution. *This problem is solved similarly to how the future value of an annuity was, except this time, instead of finding the future value of each deposit, we have to find the present value of each withdraw. We do this because we want to only deposit enough money to be able to withdraw the \$2000 at the specified time. We can collect these again in a table:*

Withdraw	Term Withdrawn	Number of times Compounded	Present Value
\$2000	1	1	$\$2000\left(1+\frac{0.06}{2}\right)^{-1} = \$2000(1.03)^{-1}$
\$2000	2	2	$\$2000\left(1+\frac{0.06}{2}\right)^{-2} = \$2000(1.03)^{-2}$
\$2000	3	3	$\$2000\left(1+\frac{0.06}{2}\right)^{-3} = \$2000(1.03)^{-3}$
\$2000	4	4	$\$2000\left(1+\frac{0.06}{2}\right)^{-4} = \$2000(1.03)^{-4}$

So adding up the present values of all these will give us the amount of money we should deposit into the account now

$$D = \$2000(1.03)^{-1} + \$2000(1.03)^{-2} + \$2000(1.03)^{-3} + \$2000(1.03)^{-4} = \$7434.20$$

Of course, just as with finding the future value of an annuity, since these fit into a pattern, we can find a formula for it; and we actually do it in the exact same way as before by computing $1.03D - D$:

$$\begin{array}{rlllll} 1.03D = & \$2000(1.03)^{0} & +\cancel{\$2000(1.03)^{-1}} & +\cancel{\$2000(1.03)^{-2}} & +\cancel{\$2000(1.03)^{-3}} & \\ -D = & & -\cancel{\$2000(1.03)^{-1}} & -\cancel{\$2000(1.03)^{-2}} & -\cancel{\$2000(1.03)^{-3}} & -\$2000(1.03)^{-4} \end{array}$$

This gives

$$1.03D - D = 0.03D = \$2,000 - \$2000(1.03)^{-4}$$

and solving for D

$$D = \$2,000\frac{1-(1.03)^{-4}}{0.03} = \$2,000\frac{1-\left(1+\frac{0.06}{2}\right)^{-4}}{0.06/2}.$$

This gives rise to the following formula

Definition 1 (Present Value of an Ordinary Annuity)**.**

$$PV = PMT\frac{1-(1+i)^{-n}}{i}$$

where

$$\begin{aligned} PV &= \textit{present value} \\ PMT &= \textit{periodic payment} \\ i &= \textit{rate per period} \\ n &= \textit{number of payments (periods)} \end{aligned}$$

Note that the payments are made at the end of each period.

In the above formula, $i = \dfrac{r}{m}$, where r is the interest rate (as a decimal) and m is the number compounding periods per year and $n = mt$ where t is the length of time of the annuity. We can rewrite the formula with r and m instead of i

$$PV = PMT\frac{1-\left(1+\frac{r}{m}\right)^{-n}}{r/m}$$

Example 2. *How much should you deposit in an account paying* 8% *compounded quarterly in order to receive quarterly payments of* \$1,000 *for the next* 4 *years?*

Solution. \$13,577.71

An interesting application of this in conjunction with sinking funds is saving for retirement.

Example 3. *The full retirement age in the US is* 67 *for people born in* 1960 *or later. Suppose you start saving for retirement at* 27 *years old and you would like to save enough to withdraw* \$40,000 *per year for the next* 20 *years. If you find a retirement savings account (for example, a Roth IRA) which pays* 4% *interest compounded annually, how much will you have to deposit per year from age* 27 *until you retire in order to be able to make your desired withdraws?*

Solution. *First, we should figure out how much money we need to have in the account at the time we retire in order to be able to make the withdraws each year. For this situation, we have*

$$PMT = \$40,000, \quad m = 1, \quad r = 0.04, \quad t = 20, \quad n = mt = 20$$

and so the present value of the retirement account at the time of retirement needs to be

$$PV = \$40,000\frac{1-\left(1+\frac{0.04}{1}\right)^{-20}}{0.04/1} = \$543,613.05.$$

So, now that we know how much we need to have in the account at the time of retirement, we can figure out how much we need to deposit into the savings account per year in order to achieve that amount in the 40 *years we have to save. To do this, we use the sinking fund formula. The future value here is the value we want at the time of retirement, so*

$$FV = \$543,613.05$$

and the other numbers are

$$r = 0.04, \quad m = 1, \quad t = 40, \quad n = mt = 40$$

Plugging this in the formula gives

$$PMT = \$543,613.05\frac{0.04/1}{\left(1+\frac{0.04}{1}\right)^{40}-1} = \$5,720.71.$$

So we will have to deposit $\$5,720.71$ *per year from age* 27 *until retirement into this account in order to be able to withdraw* $\$40,000$ *per year for* 20 *years.*